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Date 30-08-2024 Assignment #1 : Linear Algebra

Question # 1

i) $A = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & -1 \\ 2 & -2 & 1 \end{pmatrix}$

Sol

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & -1 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} \lambda-4 & -1 & -2 \\ 0 & \lambda-3 & 1 \\ -2 & 2 & \lambda-1 \end{bmatrix}$$

$$|\lambda I - A| = (\lambda-4)\left[(\lambda-3)(\lambda-1) - 2\right] + 1(0+2) - 2(0+2\lambda-6)$$

$$= (\lambda-4)\left[\lambda^2 - \lambda - 3\lambda + 3 - 2\right] + 2 - 4\lambda + 12$$

$$= (\lambda-4)(\lambda^2 - 4\lambda + 1) - 4\lambda + 14$$

$$= \lambda^3 - 4\lambda^2 + \lambda - 4\lambda^2 + 16\lambda - 4 - 4\lambda + 14$$

$$= \lambda^3 - 8\lambda^2 + 13\lambda + 10.$$

put '5' in place of 'λ'

$$= (5)^3 - 8(5)^2 + 13(5) + 10.$$

$$125 - 200 + 65 + 10 = 0$$

Using synthetic division

5	1	-8	13	10
		5	-15	-10
	1	-3	-2	0

$$\lambda^2 - 3\lambda - 2 = 0$$

Using formula $\dfrac{-b \pm \sqrt{(b^2) - 4ac}}{2a}$

$$= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-2)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{9+8}}{2} = \frac{3 \pm \sqrt{17}}{2}$$

Eigen values are $\left\{5, \frac{3+\sqrt{17}}{2}, \frac{3-\sqrt{17}}{2}\right\}$ Ans

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ii) $B = \begin{bmatrix} 5 & 2 & 1 \\ 0 & 6 & 2 \\ 3 & 1 & 4 \end{bmatrix}$

Sol

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 5 & 2 & 1 \\ 0 & 6 & 2 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} \lambda-5 & -2 & -1 \\ 0 & \lambda-6 & -2 \\ -3 & -1 & \lambda-4 \end{bmatrix}$$

$$|\lambda I - A| = (\lambda-5)[(\lambda-6)(\lambda-4)-2] + 2(0-6) - 1(0-(-3\lambda+18))$$

$$= (\lambda-5)[\lambda^2 - \cancel{4\lambda} - 6\lambda + 24 - 2] - 12 - 1(3\lambda - 18)$$

$$= (\lambda-5)(\lambda^2 - 10\lambda + 22) - 12 - 3\lambda + 18$$

$$= \lambda^3 - 10\lambda^2 + 22\lambda - 5\lambda^2 + 50\lambda - 110 - 3\lambda + 6$$

$$= \lambda^3 - 15\lambda^2 + 69\lambda - 104$$

Put '8' in place of 'λ'

$$(8)^3 - 15(8)^2 + 69(8) - 104$$

$$512 - 960 + 552 - 104 \boxed{= 0}$$

Using Synthetic division

$$\begin{array}{r|rrrr} 8 & 1 & -15 & 69 & -104 \\ & & 8 & -56 & 104 \\ \hline & 1 & -7 & 13 & 0 \end{array}$$

$$\lambda^2 - 7\lambda + 13 = 0$$

Using formula $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{7 \pm \sqrt{-3}}{2}$$

Eigen values are: $\left\{ 8, \dfrac{7+\sqrt{-3}}{2}, \dfrac{7-\sqrt{3}}{2} \right\}$ ans

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Question #2:

1) $\frac{dy}{dx} = 8x^3y^2$

Sol

$\int \frac{1}{y^2} dy = \int 8x^3 dx$

$\int y^{-2} dy = \int 8x^3 dx$

$\boxed{\frac{1}{y} = 2x^4 + c} \Rightarrow \boxed{y = \frac{1}{2x^4 + c}}$ ans

2) $\frac{dy}{dx} = 1 + y^2$

Sol

$\frac{1}{1+y^2} dy = dx$

$\int \frac{1}{1+y^2} dy = \int dx$

$\tan^{-1}(y) = x + c$

$\boxed{y = \tan(x+c)}$ ans

3) $\frac{dy}{dx} = \frac{x(x^2+1)}{4y^3}$, $y(0) = -\frac{1}{\sqrt{2}}$

Sol

$\int 4y^3 dy = \int x(x^2+1) dx \Rightarrow \int 4y^3 dy = \int (x^3+x) dx$

$\cancel{\boxed{y^4 = \frac{(x^2+1)^2}{2} + c}}$ — $\boxed{y^4 = \frac{x^4}{4} + \frac{x^2}{2} + c}$ — ①

put $y(0) = -\frac{1}{\sqrt{2}}$

$\cancel{\left(-\frac{1}{\sqrt{2}}\right)^4 = \frac{(0+1)^2}{2} + c}$ $\left(-\frac{1}{\sqrt{2}}\right)^4 = \frac{(0)^4}{4} + \frac{(0)^2}{2} + c$

$\frac{1}{4} = 0 + c$

$\cancel{\frac{1}{4} - \frac{1}{2} = c} \Rightarrow \boxed{c = \frac{1}{4}}$

put the value of 'c' in eq ①

① $\Rightarrow$ $\cancel{\boxed{y^4 = \frac{(x^2+1)^2}{2} + \frac{1}{4}}}$ ~~ans~~ ① $\Rightarrow \boxed{y^4 = \frac{x^4}{4} + \frac{x^2}{2} + \frac{1}{4}}$

ans

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Question #3 (Homogenous Differential Equation)

1) $(x-y)dx + (x+y)dy = 0$

Sol

$$\cancel{(x-y)dx = -(x+y)dy}$$

$$\cancel{\int (x-y)\,dx = \int -(x+y)\,dy}$$

$$(x-y)dx = -(x+y)dy$$

$$\frac{dy}{dx} = \frac{-(x-y)}{x+y}$$

$$\frac{dy}{dx} = \frac{y-x}{x+y} \quad \text{--- } (1)$$

let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$(1) \Rightarrow \quad v + x\frac{dv}{dx} = \frac{vx - x}{x + vx}$$

$$x\frac{dv}{dx} = \frac{\cancel{x}(v-1)}{\cancel{x}(1+v)} - v$$

$$x\frac{dv}{dx} = \frac{v-1}{1+v} - v$$

$$x\frac{dv}{dx} = \frac{v-1-v-v^2}{1+v}$$

$$x\frac{dv}{dx} = \frac{-1-v^2}{1+v}$$

$$\frac{1+v}{(1+v^2)}\,dv = -\frac{1}{x}\,dx$$

$$\int \frac{dv}{1+v^2} + \frac{1}{2}\int \frac{2v}{1+v^2}\,dv = -\int \frac{1}{x}\,dx$$

$$\tan^{-1}(v) + \frac{\ln(1+v^2)}{2} = -\ln x + c$$

$$\tan^{-1}(y/x) + \frac{\ln\left(1+(y/x)^2\right)}{2} = -\ln x + c$$

$$\boxed{\cancel{\tan^{-1}\left(\frac{x^2+y^2}{x^2}\right) + \frac{\ln\left(\frac{x^2+y^2}{x^2}\right)}{2} = -\ln x + c}}$$ ans

$$\tan^{-1}(y/x) + \ln\left(\frac{x^2+y^2}{x^2}\right)^{1/2} = -\ln x + c$$

$$\tan^{-1}(y/x) + \ln(x^2+y^2)^{1/2} - \ln(x^2)^{1/2} = -\ln x + c$$

$$\boxed{\tan^{-1}(y/x) + \ln(x^2+y^2)^{1/2} = c}$$ ans

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2) $(x^3+y^2\sqrt{x^2+y^2})\,dx - xy\sqrt{x^2+y^2}\,dy = 0$

Sol

$(x^3+y^2\sqrt{x^2+y^2})\,dx = xy\sqrt{x^2+y^2}\,dy$

$\dfrac{x^3+y^2\sqrt{x^2+y^2}}{xy\sqrt{x^2+y^2}} = \dfrac{dy}{dx}$

$y = vx,\quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$v + x\dfrac{dv}{dx} = \dfrac{x^3+(vx)^2\sqrt{x^2+v^2x^2}}{x(vx)\sqrt{x^2+v^2x^2}} = \dfrac{x^3+v^2x^2\sqrt{x^2(1+v^2)}}{vx^2\sqrt{x^2(1+v^2)}} = \dfrac{x^3+v^2x^3\sqrt{1+v^2}}{vx^3\sqrt{1+v^2}}$

$v + x\dfrac{dv}{dx} = \dfrac{1+v^2\sqrt{1+v^2}}{v\sqrt{1+v^2}} \Rightarrow x\dfrac{dv}{dx} = \dfrac{(1+v^2\sqrt{1+v^2}) - v^2\sqrt{1+v^2}}{v\sqrt{1+v^2}}$

$x\dfrac{dv}{dx} = \dfrac{1}{v\sqrt{1+v^2}} \Rightarrow v\sqrt{1+v^2}\,dv = \dfrac{1}{x}\,dx$

$\dfrac{1}{2}\int 2v\sqrt{1+v^2}\,dv = \int \dfrac{1}{x}\,dx$

$\dfrac{1}{2}\dfrac{(1+v^2)^{3/2}}{3/2} = \ln x + c_1$

$\dfrac{1}{3}(1+v^2)^{3/2} = \ln x + c$

$\dfrac{1}{3}(1+y^2/x^2)^{3/2} = \ln x + c_1$

$\dfrac{1}{3}\left(\dfrac{x^2+y^2}{x^2}\right)^{3/2} = \ln x + c_1$

$\dfrac{1}{3x^3}(x^2+y^2)^{3/2} = \ln x + c_1$

$(x^2+y^2)^{3/2} = 3x^3\ln x + 3x^3c_1$

let $3c_1 = c_2$

$\boxed{(x^2+y^2)^{3/2} = 3x^3\ln x + c_2x^3}$ an

3) $(y+\sqrt{x^2+y^2})\,dx - x\,dy = 0 \quad y(1)=0$

Sol

$y+\sqrt{x^2+y^2}\,dx = x\,dy$

$dy/dx = \dfrac{y+\sqrt{x^2+y^2}}{x}$

let $y = vx,\quad dy/dx = v + x\dfrac{dv}{dx}$

$v + x\dfrac{dv}{dx} = \dfrac{vx+\sqrt{x^2+(v^2x^2)}}{x}$

$v + x\dfrac{dv}{dx} = \dfrac{vx + x\sqrt{1+v^2}}{x}$

$v + x\dfrac{dv}{dx} = \dfrac{\cancel{x}(v+\sqrt{1+v^2})}{\cancel{x}}$

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$$x\frac{dv}{dx} = \cancel{v} + \sqrt{1+v^2} - \cancel{v}$$

$$\int \frac{1}{\sqrt{1+v^2}}\,dv = \int \frac{1}{x}\,dx$$

~~let $u = 1+x^2$~~

~~$du = 2x\,dx$~~

~~Multiply and Divide by 2~~

~~$\frac{1}{2}\int$~~

$$\ln\left(v+\sqrt{1+v^2}\right) = \ln x + \ln c$$

$$\ln\left(y/x + \sqrt{1+y^2/x^2}\right) = \ln(x) + \ln c$$

$$\ln\left(y/x + \sqrt{\frac{x^2+y^2}{x^2}}\right) = \ln(x) + \ln c$$

$$\boxed{\ln\left(\frac{y+\sqrt{x^2+y^2}}{x^2}\right) = \ln(xc)}$$

Apply 'e' on BS

$$\frac{y+\sqrt{x^2+y^2}}{x^2} = xc$$

$$\boxed{y+\sqrt{x^2+y^2} = x^3c}$$

put $y(1)=0$ in above equation

$$0+\sqrt{1^2+0^2} = (1)^3c$$

$$\boxed{1 = c}$$

put the value of c

$$y+\sqrt{x^2+y^2} = x^3(1)$$

$$\boxed{y = x^3 - \sqrt{x^2+y^2}}$$ ans

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2) $\frac{dy}{dx} = \frac{x-2y+5}{2x+y-1}$

Sol

Let $x = X + h$; $\frac{dx}{dX} = 1 \Rightarrow \boxed{dx = dX}$

$y = Y + k$; $\frac{dy}{dY} = 1 \Rightarrow \boxed{dy = dY}$

$$\frac{dY}{dX} = \frac{(X+h) - 2(Y+k) + 5}{2(X+h) + (Y+k) - 1}$$

$$\frac{dY}{dX} = \frac{X+h-2Y-2k+5}{2X+2h+Y+k-1} \Rightarrow \boxed{\frac{dY}{dX} = \frac{X-2Y+h-2k+5}{2X+Y+2h+k-1}} \quad (1)$$

Let,

$h - 2k + 5 = 0$ — (A)

$2h + k - 1 = 0$ — (B)

Multiply eq (A) by '2' and Subtract from eq (B)

$$\begin{array}{r} 2h - 4k + 10 = 0 \\ -2h \mp k \pm 1 = 0 \\ \hline -5k + 11 = 0 \end{array}$$

$k = \frac{-11}{-5} \Rightarrow \boxed{k = 11/5}$

$\cancel{\boxed{k = 9/5}}$

put $k = 11/5$ in eq (B)

~~$2h + 9/5 - 1 = 0$~~

~~$h = (4/5)/2$~~

~~$\boxed{h = 2/5}$ $\boxed{h = 2/5}$~~

$2h + \frac{11}{5} - 1 = 0$

$h = \frac{-6/5}{2} \Rightarrow \boxed{h = -3/5}$

put the values in eq (1)

(1) $\Rightarrow$ ~~$\frac{X - 2Y + (-2) - 2(3) + 5}{2X - Y + 2(-2) - (3) - 1}$~~ $\Rightarrow$ ~~$\frac{X - 2Y + 2/5 - 2(9/5) + 5}{2X + Y + 2(4/5) + 9/5 - 1}$~~

$$(1) \Rightarrow \frac{dY}{dX} = \frac{X - 2Y + (-3/5) - 2(11/5) + 5}{2X + Y + 2(-3/5) + (11/5) - 1}$$

$$\frac{dY}{dX} = \frac{X - 2Y}{2X + Y}$$

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$$\frac{dY}{dX} = \frac{X-2Y}{2X+Y}$$

let $Y = VX$

$$dY/dX = V + X\frac{dV}{dX}$$

$$V + X\frac{dV}{dX} = \frac{X-2VX}{2X+VX}$$

$$V + X\frac{dV}{dX} = \frac{\cancel{X}(1-2V)}{\cancel{X}(2+V)}$$

$$X\frac{dV}{dX} = \frac{1-2V-2V+V^2}{2+V}$$

$$X\frac{dV}{dX} = \frac{1-4V-V^2}{2+V}$$

$$\left(\frac{2+V}{1-4V-V^2}\right)dV = \frac{1}{X}dX$$

Multiply and divide by '-2'

$$-\frac{1}{2}\int\left(\frac{-4-2V}{1-4V-V^2}\right)dV = \int\frac{1}{X}dX$$

$$-\frac{1}{2}\ln(1-4V-V^2) = \ln X + c$$

$$\boxed{-\frac{1}{2}\ln\left(1-4(Y/X) - Y^2/X^2\right) = \ln X + c} \quad -②$$

As we know $x = X - 3/5 \Rightarrow \boxed{X = x + 3/5}$

$y = Y + 11/5 \Rightarrow \boxed{Y = y - 11/5}$

put the above values in eq ②

$$-\frac{1}{2}\ln\left(1 - \frac{4(y-11/5)}{x+3/5} - \left(\frac{y-11/5}{x+3/5}\right)^2\right) = \ln(x+3/5) + c$$

$$\ln\left(\frac{\frac{5x+3}{5} - \frac{4y+44/5}{}}{\frac{5x+3}{5}} - \left(\frac{\frac{5y-11}{\cancel{5}}}{\frac{5x+3}{\cancel{5}}}\right)^2\right) = \ln\left(\frac{5x+3}{5}\right) + c$$

$$\boxed{\ln\left[\frac{\frac{5x+3}{5} - \frac{20y+44}{5}}{\frac{5x+3}{5}} - \left(\frac{5y-11}{5x+3}\right)^2\right] = \ln(5x+3) - \ln(5) + c}$$

ans

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Question #5

$(4x^3 e^{x+y} + x^4 e^{x+y} + 2x)\,dx + (x^4 e^{x+y} + 2y)\,dy = 0, \quad y(0) = 1$

Sol

Let $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$M = 4x^3 e^{x+y} + x^4 e^{x+y} + 2x, \quad N = x^4 e^{x+y} + 2y$

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(4x^3 e^{x+y} + x^4 e^{x+y} + 2x), \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^4 e^{x+y} + 2y)$

$\boxed{\frac{\partial M}{\partial y} = 4x^3 e^{x+y} + x^4 e^{x+y}}$ — 1 , $\boxed{\frac{\partial N}{\partial x} = x^4 e^{x+y} + 4x^3 e^{x+y}}$ — 2

Let,

$\frac{\partial f}{\partial y} = N(x,y), \quad \frac{\partial f}{\partial x} = M(x,y)$

$\boxed{\frac{\partial f}{\partial y} = (x^4 e^{x+y} + 2y)}$ — 3 , $\boxed{\frac{\partial f}{\partial x} = 4x^3 e^{x+y} + x^4 e^{x+y} + 2x}$ — 4

Integrate eq 3

$\int \partial f = \int (x^4 e^{x+y} + 2y)\,dy$

$\boxed{f = x^4 e^{x+y} + y^2 + h(x)}$ — 5

Taking partial derivative on 'f' wrt 'x'

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^4 e^{x+y} + y^2 + h(x))$

$\boxed{\frac{\partial f}{\partial x} = x^4 e^{x+y} + e^{x+y}(4x^3) + \frac{\partial}{\partial x} h(x)}$

put the value of $\partial f/\partial x$ in eq 4

4 ⇒ $\cancel{x^4 e^{x+y}} + \cancel{4x^3 e^{x+y}} + \partial/\partial x\, h(x) = \cancel{4x^3 e^{x+y}} + \cancel{x^4 e^{x+y}} + 2x$

$\partial/\partial x (h(x)) = 2x$

Taking Integration

$\int \partial h(x) = \int 2x\,dx$

$\boxed{h(x) = x^2 + c}$

put the value of $h(x)$ in eq 5

$\boxed{f = x^4 e^{x+y} + y^2 + x^2 + c}$ — 6

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Now put $y(0) = 1$

$f(0,1) = (0)^4 e^{0+1} + (1)^2 + (0)^2 + c$

$0 = 0 + 1 + 0 + c$

$\boxed{c = -1}$

put value of 'c' in eq (6)

$\boxed{f = x^4 e^{x+y} + y^2 + x^2 - 1}$ ans

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